

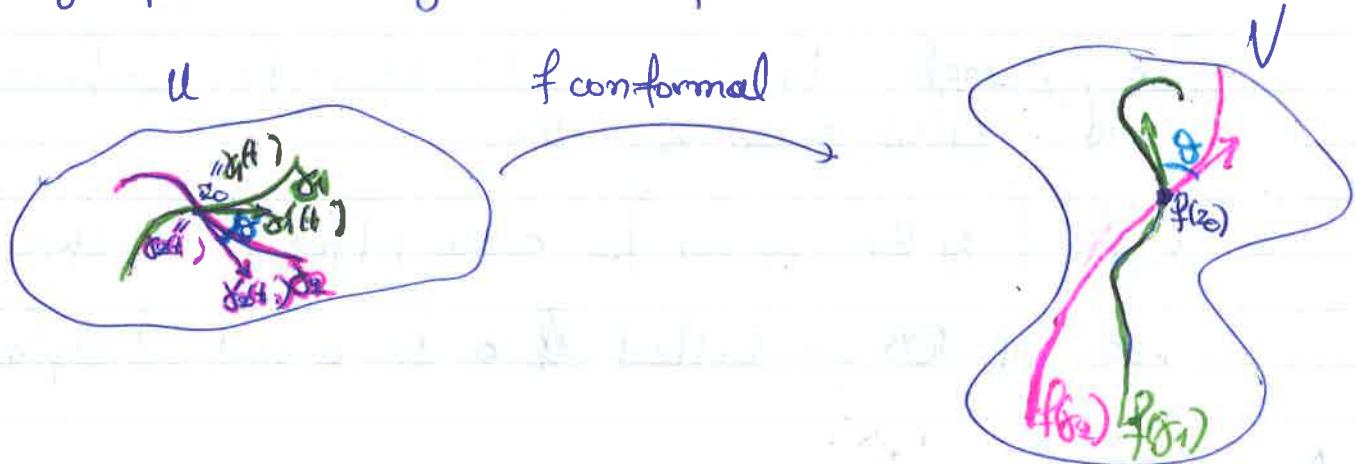
(1)

→ Conformal mapping in 2-dim fluid mechanics:

→ Def: Let $f: \overset{\text{open}}{\mathcal{U}} \rightarrow \mathbb{C}$ be holomorphic.

We say that f is conformal at $z_0 \in \mathcal{U}$ if $f'(z_0) \neq 0$.

→ The amazing property of conformal mappings is that they preserve angles. In particular, let



Let $\gamma_1, \gamma_2: [a, b] \rightarrow \mathcal{U}$ be two curves, that meet at $z_0 \in \mathcal{U}$ (i.e., $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$, for $t_1, t_2 \in [a, b]$). Let θ be the angle between the tangents of γ_1 and γ_2 at z_0 (i.e., the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$).

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Then, the images $f(\gamma_1)$ and $f(\gamma_2)$ of the two curves through f ($f \circ \gamma_1 : [a, b] \rightarrow V$, $f \circ \gamma_2 : [a, b] \rightarrow V$),

which obviously pass through $f(z_0)$ ($z_0 = f \circ \gamma_1(t_1) = f \circ \gamma_2(t_2)$),

also

have tangents at $f(z_0)$ with angle δ between them (just like in the preimage of f).

In other words, when f is conformal and $\gamma_1(t_1) = \gamma_2(t_2) = z_0$, then

angle $(\gamma_1'(t_1), \gamma_2'(t_2)) = \text{angle}((f \circ \gamma_1)'(t_1), (f \circ \gamma_2)'(t_2))$
 (see p. 709 in textbook for a sort of explanation).



If $f: U \rightarrow V$ is conformal,

then also $f^{-1}: V \rightarrow U$ is conformal

(because $(f^{-1} \circ f)'(z_0) = (f^{-1})'(f(z_0)) \cdot f'(z_0)$, $\forall z_0 \in U$,
 so $(f^{-1})'(f(z_0)) = \frac{(\text{Id})'(z_0)}{f'(z_0)} \underset{\substack{\text{Id}: \mathbb{C} \rightarrow \mathbb{C} \\ z \mapsto z}}{=} \frac{1}{f'(z_0)} \neq 0$,
 $\forall f(z_0) \in V$.)

for that reason, conformal mappings can be very useful when solving partial differential equations in 2-dim domains.

As an example, we will consider the following general problem in fluid mechanics:

Suppose that we have a fluid mechanics problem on a 2-dim domain U , such that the fluid at any $x \in U$ has velocity $\vec{v}(x)$. Suppose the flow is:

incompressible

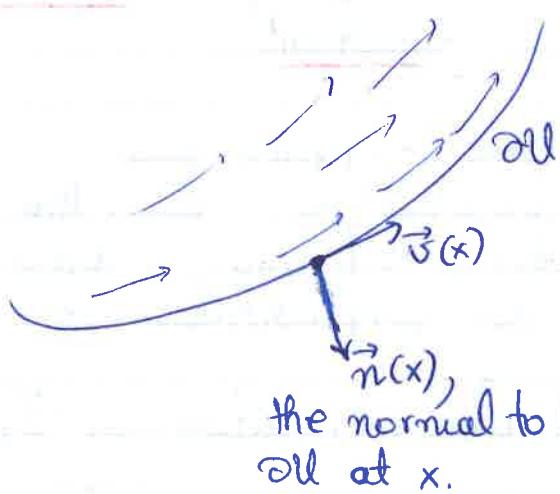
$$\left. \begin{aligned} &\text{i.e., } \operatorname{div} \vec{v} = 0, \\ &\text{i.e., } \nabla \cdot \vec{v} = 0, \\ &\text{i.e., } \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \end{aligned} \right\} \text{on } U$$

irrotational

$$\left. \begin{aligned} &\text{i.e., } \operatorname{curl} \vec{v} = 0, \\ &\text{i.e., } \nabla \times \vec{v} = 0, \\ &\text{i.e., } \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \end{aligned} \right\} \text{on } U$$

steady

i.e., there is no time dependence; at any time, the particle at $x \in U$ will have velocity $\vec{v}(x)$ independent of t



and we have the boundary condition that

$$\vec{v}(x) \cdot \vec{n}(x) = 0, \quad \forall x \in \partial U.$$

i.e., $\vec{v}(x)$ is tangent to ∂U $\forall x \in \partial U$.

→ What is $\vec{v}(x)$ $\forall x \in U$?

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If U has no holes, the fact that the flow is irrotational actually implies that there exists some

$$\phi : U \rightarrow \mathbb{R}$$

known as the velocity potential.

$$\text{s.t. } \vec{v}(x) = (\nabla \phi)(x), \quad \forall x \in U$$

(i.e., the desired velocity is the gradient of some function).

We go to the conditions in the previous page, and we plug in $\nabla \phi$ in the place of \vec{v}

(in all the conditions apart $\nabla \times \vec{v} = 0$, as we have used that already).

We need to find $\phi : U \rightarrow \mathbb{R}$, s.t.

(i) $\vec{\nabla} \cdot \vec{\nabla} \phi = 0 \Leftrightarrow \Delta \phi = 0$, i.e. ϕ satisfies Laplace's equation!

(ii) $\underbrace{\vec{\nabla} \phi(x)}_{\text{i.e., } \phi \text{ doesn't change in the direction of the boundary.}} \cdot \vec{n}(x) = 0, \quad \forall x \in \partial U$ ~ this is really just a condition on angles: the vectors $\vec{\nabla} \phi$ and \vec{n} have to be perpendicular on ∂U .

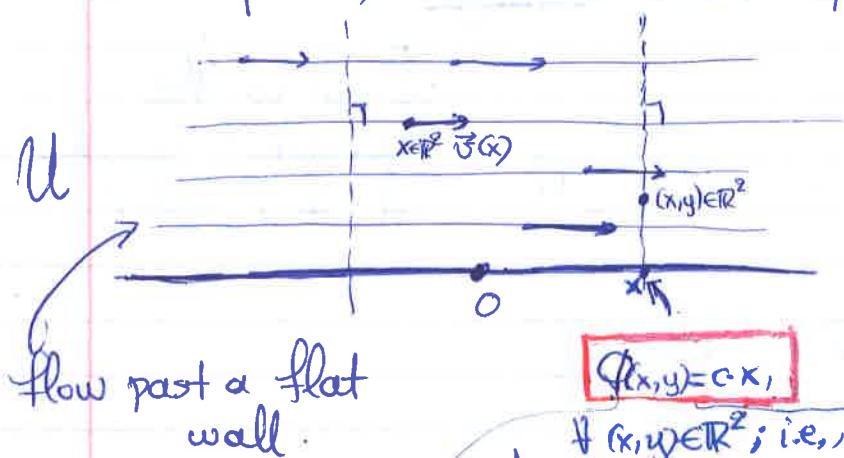
So, we just need to solve Laplace's equation on U , and then take the gradient of the solution; this will be \vec{v} !

→ What if U was the upper half-plane?

(5)

People know how to solve this problem in the upper half-plane. It is known that, if $U = \text{upper half-plane}$ at constant speed

half-plane, then



the flow moves

along all horizontal lines (streamlines).

More precisely:

Φ is constant along any vertical line, and it changes linearly in the horizontal direction.

(Thus, $\vec{v} = \nabla \Phi$, which is perpendicular to $\{\Phi = \text{const}\}$, is horizontal, and constant).

$$\Phi(x, y) = cx,$$

$\Phi(x, y) \in \mathbb{R}^2$; i.e., Φ is const. on every vertical line and changes linearly in the horizontal direction.

Notice indeed how conditions (i) and (ii) are satisfied

for this $\vec{v}(x) = \nabla \Phi$, for Φ as above.

→ Here, $\nabla \Phi + \{ \Phi = \text{constant} \}$ is obvious ($\{ \Phi = \text{const} \}$ is a vertical line, so on it $\frac{\partial \Phi}{\partial y} = 0$, thus $\nabla \Phi$ is horizontal). But this would hold even if $\{ \Phi = \text{const} \}$ was a curve (by the implicit function theorem). So,

$\vec{v}(x) = \nabla \Phi(x)$ is always perpendicular to $\{ \Phi = \text{const} \}$. Another perspective: Since Φ satisfies Laplace's equation, it has a harmonic conjugate Ψ ,

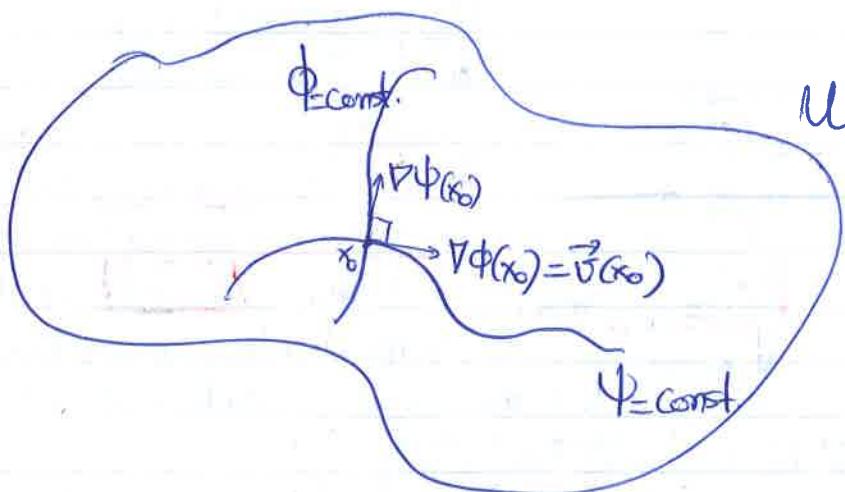
and, as it happens for harmonic conjugates (by Cauchy-Riemann conditions for holomorphic $\Phi + i\Psi$), we have

that the curves $\{ \Phi = \text{constant} \}$ and $\{ \Psi = \text{constant} \}$ are perpendicular when they meet. And $\{ \Psi = \text{constant} \}$ are exactly the streamlines (it can be proved)!

(5)

→ Many of the above hold for $\phi : U \rightarrow \mathbb{R}$
 that satisfies $\nabla \phi = 0$ on U , no matter
 how crazy U looks!

In particular, always:



If ψ is the harmonic conjugate of ϕ , then:

- $\forall x_0 \in U$, the curves $\{\phi=\text{const}\}$ and $\{\psi=\text{const}\}$ are perpendicular when they meet (i.e. $\nabla \psi(x_0) \cdot \nabla \phi(x_0) = 0$),

as $\nabla \psi(x_0)$ is always perpendicular to $\{\psi=\text{const}\}$ and $\nabla \phi(x_0)$ " " " "

- Clearly, by the above we have that

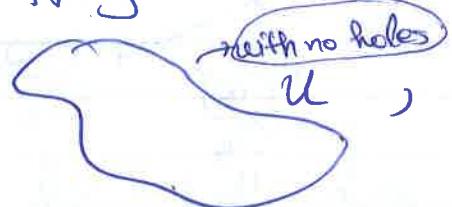
$\nabla \phi(x_0) = \vec{v}(x_0)$ is tangent $\overset{\text{at } x_0}{\text{to}} \{\psi=\text{const}\}$

as this is always (when $\{\psi=\text{const}\}$ passes through x_0).
 perpendicular to $\{\phi=\text{const}\}$,
 by implicit function theorem

(7)

- But, even better, $\{\Psi = \text{const}\}$ is the path a particle at x_0 will follow in this flow! That is why the curves $\{\Psi = \text{const}\}$ are called streamlines: they are the paths the fluid particles follow during the flow.

So, already complex analysis (the existence of harmonic conjugates and the Cauchy-Riemann conditions) has revealed a lot. Conformal mappings will now help us to take any



change it ^{conformally} into the upper half-plane, where we know what happens with incompressible, irrotational flows, and then transfer the solution back to U .

This is based on a simple observation and an important theorem:

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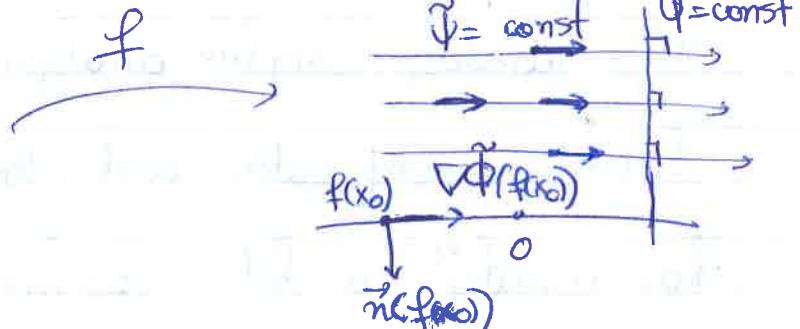
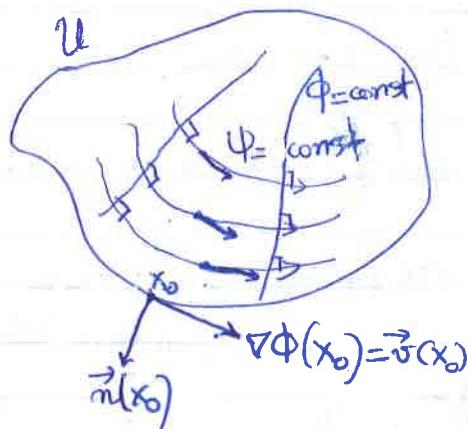
Let $\Phi: U \rightarrow \mathbb{R}$, with

$$(i) \Delta \Phi = 0 \text{ on } U$$

and (ii) $\nabla \Phi(x) \cdot \vec{n}(x) = 0, \forall x \in \partial U$. What is $\nabla \Phi$?



Observation: Suppose that we can find a conformal mapping f that sends U to the upper half-plane,



such that all ∂ (upper half-plane)
 $=$ real line.

If $\tilde{\Phi}: (\text{upper half-plane}) \rightarrow \mathbb{R}$

satisfies $\Delta \tilde{\Phi} = 0$ on upper half-plane

and $\nabla \tilde{\Phi}(x) \cdot \vec{n}(x) = 0 \quad \forall x \text{ on } \partial(\text{upper half-plane}),$

* just replace $\overset{(x,y)}{z}$ by $f(z)$ in formula $\tilde{\Phi}(x,y) = c \cdot x$.

then, $\Phi := \tilde{\Phi} \circ f$, (i.e. $\Phi(y) = \tilde{\Phi}(f(y)) \quad \forall y \in U$)

satisfies conditions (i) and (ii)

! Problem 4,
p. 716

(3)

So, all we need is to find such f ; then,

Φ will be $\begin{matrix} \downarrow \\ f \circ \phi \end{matrix}$. And then, $\boxed{\vec{v} = \nabla \Phi}$.

→ Notice that, since conformal mappings preserve angles, we have that, since $\nabla \tilde{\Phi}$ is perpendicular to the real line, then also $\nabla \Phi$ is perpendicular to ∂U (which is why (ii) is satisfied).

→ Also, $\{\tilde{\Phi} = \text{const}\}$ maps to $\{\Phi = \text{const}\}$ via f^{-1} ; similarly for $\tilde{\Psi}$. So, the streamlines $\{\Psi = \text{const}\}$ in U are just the horizontal lines, the inverse images of the streamlines via f ! I.e., to find the streamlines for our flow, we just

calculate f^{-1} (each horizontal line). The flow is just tangent to those streamlines.

not a surprise!
 $\nabla \Phi$ should be tangent to these curves, as f^{-1} preserves angles, and $\nabla \tilde{\Phi}$ was tangent to the horizontal lines.

we don't need to find Ψ at all.

$\nabla \Phi = \vec{v}$ to these streamlines.

We fully understand our flow by the three \star 's above.

(10)

→ finally, notice that the curves $\{\psi = \text{const}\}$
 and $\{\phi = \text{const}\}$ are perpendicular, as

ϕ and ψ are harmonic conjugates. However,

this is also verified by the fact that, since

f^{-1} is conformal, and $\{\tilde{\phi} = \text{const}\} \perp \{\tilde{\psi} = \text{const}\}$,

then $\underbrace{f^{-1}(\{\tilde{\phi} = \text{const}\})}_{\{\phi = \text{const}\}} \perp \underbrace{f^{-1}(\{\tilde{\psi} = \text{const}\})}_{\{\psi = \text{const}\}}$ as well.

Theorem: (Riemann mapping theorem)

If $U \subseteq \mathbb{C}$ is ^{open} simply connected (i.e., it has no holes),

then there exists a conformal mapping

$f: U \rightarrow$ upper half-plane.

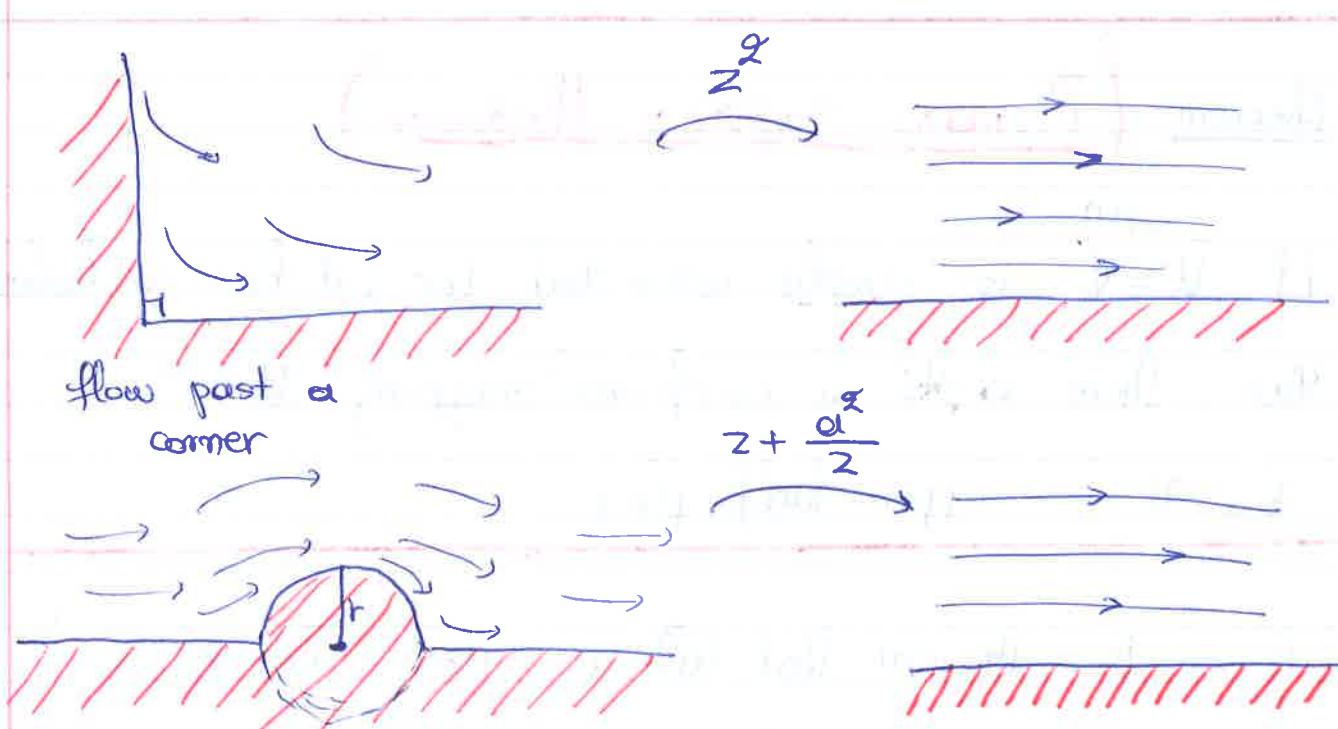
It is this theorem that tells us that what we described above is not vacuous non sense ; we know

that, no matter how crazy our U is as long as it has no holes, we can map it conformally to the upper half-plane!

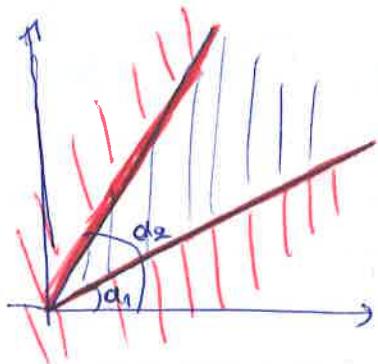
(11)

So, the method we describe above to find a flow on a general Ω will always work, as long as we can find the appropriate conformal f that sends Ω to the upper half-plane (if we can't find it, it's just due to our inability; it exists!).

Here is a chart on which holomorphic maps send some standard shapes to the upper half-plane:

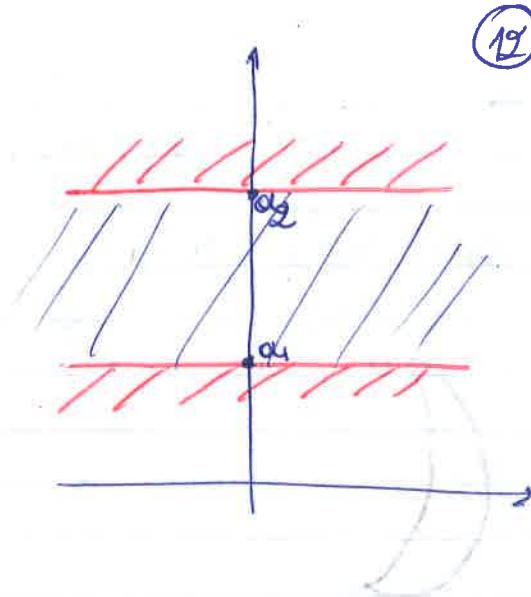


flow past a cylindrical bump



$\ln z$
Principal branch

(Figure 10.1
in p. F11
is a special
case of this).



You can also look up Möbius transformations.

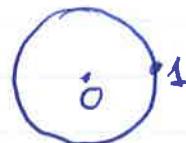
they are of the form $f: \mathbb{C} \rightarrow \mathbb{C}$,
 \downarrow
 or $\mathbb{C}\setminus\{\text{point}\}$

with

$$f(z) = \frac{az+b}{cz+d}.$$

These send (lines and circles)
 to
 (lines or circles).

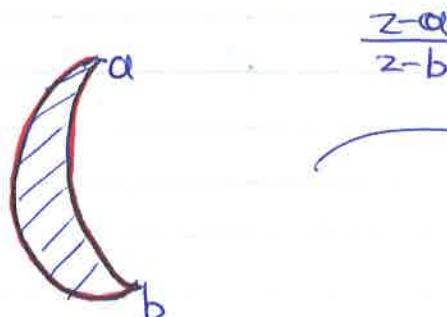
for example, the unit circle



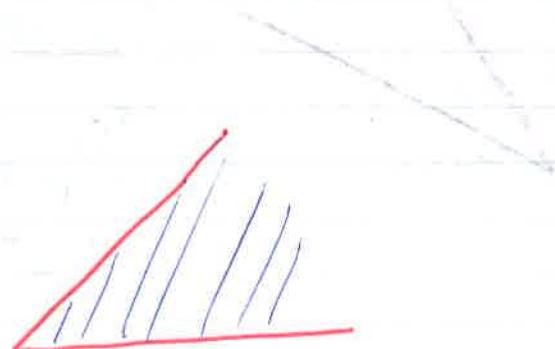
is sent to a line through $\frac{1}{z-1}$ (it can only
 be sent to sth unbounded, due to the blowing up at 1),
 while it is sent to a circle (itself actually) through $\frac{1}{z}$.

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Some more mappings, that can further lead to the upper half-plane:



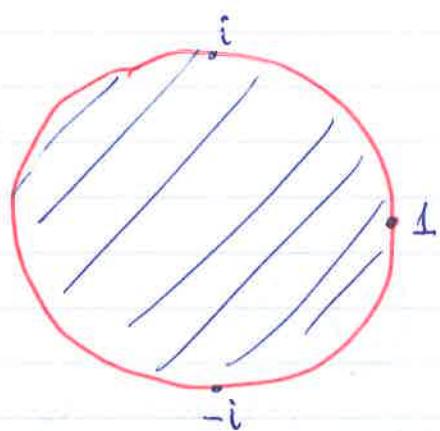
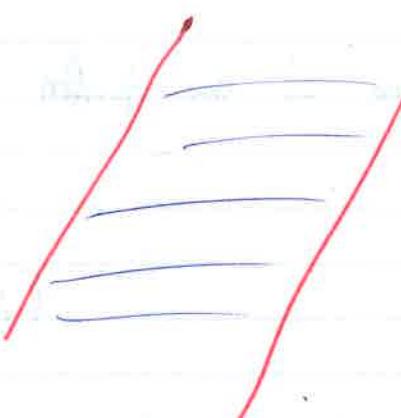
$$\frac{z-a}{z-b}$$



↑
intersection
of discs



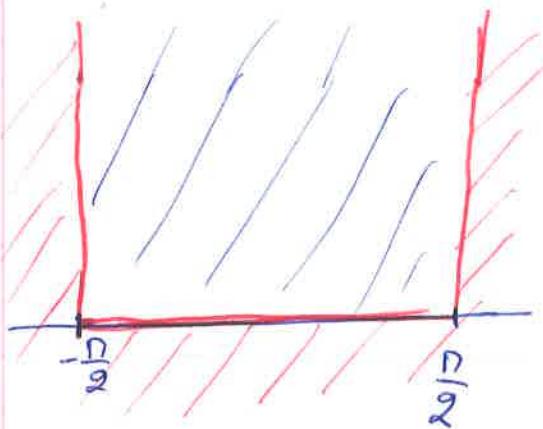
$$\frac{1}{z-a}$$



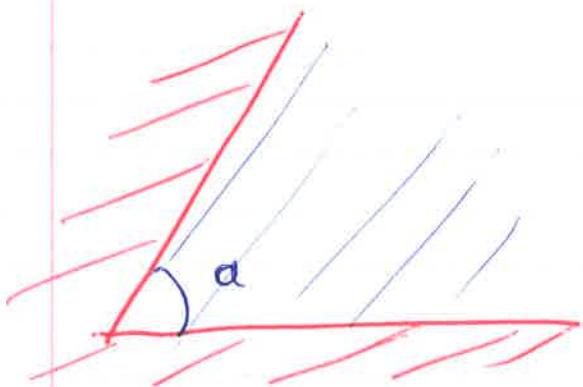
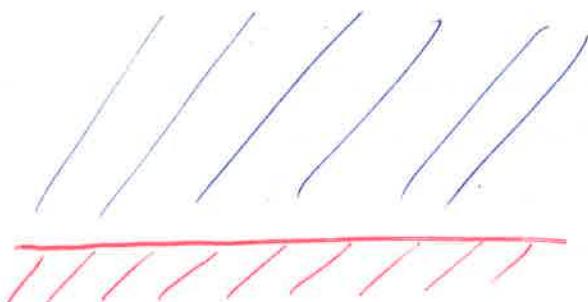
$$\frac{z-i}{z+i}$$



(14)



$\sin z$

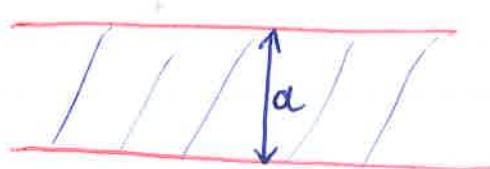


$\frac{\pi}{z^\alpha}$



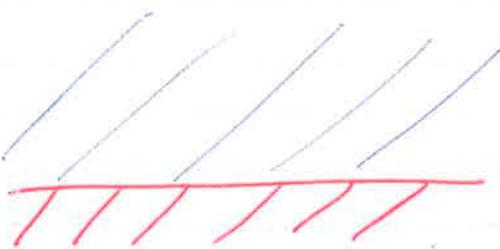
white

$\log z$



Schwarz-Christoffel
formula

Polygons



→ Of course, the above ideas can be used in any problem where we are required to find solutions ϕ to Laplace's equation on a domain without holes, with some angular boundary conditions

(i.e., $\nabla\phi$ doesn't have to be the velocity of any fluid).

A situation like this is when we want to find how heat is distributed in a domain (with insulated boundary, with no sources of heat other than particular parts of the boundary where we keep the temperature constant). In this case it is known that the temperature satisfies Laplace's equation. (See p. 711 for such an example).

If ϕ is the temperature at each point, then $\{\phi = \text{const}\}$ are the curves along which heat is constant (the "isotherms"). The

boundary condition is that $\nabla\phi$ perpendicular to the boundary (i.e. ϕ doesn't change in the direction of the boundary), so this also holds for

(16)

$\nabla \tilde{\phi}$ and the boundary of $f(U)$, for f
 \downarrow
 $\tilde{\phi} = \phi \circ f$ any holomorphic conformal map.

So, we can just transform U to
 a parallelogram, find the isothermals there (easy),
 and get their preimages via f to find
 the isothermals in the original domain U .